Formalization of Infinite Dimension Linear Spaces with Application to Quantum Theory

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Outline

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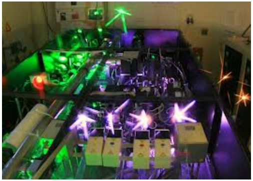
Motivation

Linear algebra.

Bioinformatics

Digital Signal Processing

Control Systems: Robotics



Quantum Optics

Related Tools

Numerical

Computer Algebra Systems

Related work

- HOL Light
  - J. Harrison 2005: Euclidean spaces $\mathbb{R}^N$.
  - S. Khan Afshar, V. Aravantinos 2012: complex vector spaces $\mathbb{C}^N$.

$\longrightarrow$ finite dimension only

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  $\longrightarrow$ finite dimension only
- PVS 2012: Real vector spaces $\mathbb{R}^N$, with application in control theory.

  $\longrightarrow$ finite dimension and real only

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  $\longrightarrow$ finite dimension only
- PVS 2012: Real vector spaces $\mathbb{R}^N$, with application in control theory.
  $\longrightarrow$ finite dimension and real only
- Coq: an abstract development of some preliminary linear algebra.
  $\longrightarrow$ not suitable for practical application,
  (missing important notions, e.g, self-adjointness)

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Quantum Mechanics

In general (quantum or classic):

A physical system is described by a state
$=$ collection of informations.

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Classical

- State $=$ collection of real variables.

Quantum

- State $=$ complex-valued functions.

Quantum Mechanics

In general (quantum or classic):

A physical system is described by a state
= collection of informations.

Classical

- State = collection of real variables.
- Measurement = deterministic.

Quantum

- State = complex-valued functions.
- Measurement = statistical.

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In general (quantum or classic):

A physical system is described by a state
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- State = collection of real variables.
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- State = complex-valued functions.
- Measurement = statistical.
- Observables = self-adjoint operators.

Quantum Mechanics

In general (quantum or classic):

A physical system is described by a state
= collection of informations.

Classical

- State = collection of real variables.
- Measurement = deterministic.
- Observables = real functions.
- Interested in measured values themselves.

Quantum

- State = complex-valued functions.
- Measurement = statistical.
- Observables = self-adjoint operators.
- Interested in expectation = eigenvalues.

Formalization

A glance of the required notions:

Definition (Quantum Space)

$\texttt{is\_qspace}\ ((\texttt{vs}, \texttt{inprod}) : \texttt{qspace}) \ \Leftrightarrow$
$\quad \texttt{is\_subspace}\ \texttt{vs} \wedge \texttt{is\_inner\_product}\ \texttt{inprod}$

Definition (Observable)

$\texttt{is\_observable}\ (\texttt{op} : \texttt{qstate} \rightarrow \texttt{qstate})\ ((\texttt{vs}, \texttt{inprod}) : \texttt{qspace}) \ \Leftrightarrow$
$\quad \texttt{is\_qspace}\ (\texttt{vs}, \texttt{inprod}) \wedge \texttt{is\_self\_adjoint}\ \texttt{op}\ \texttt{inprod} \wedge$
$\quad \forall\ \texttt{x}.\ \texttt{x} \in \texttt{vs} \Rightarrow \texttt{op}\ \texttt{x} \in \texttt{vs}$

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Complex-valued functions (1/2)

Definition (Complex functions type)

$\mathtt{cfun} = \mathtt{A} \to \mathtt{complex}$

Definition (Algebraic operations over $\mathtt{cfun}$)

Operation	Notation	Definition
cfun_add	$\mathtt{f}_1 +_{\mathtt{cfun}} \mathtt{f}_2$	$\lambda \mathtt{x} : \mathtt{A}.\ \mathtt{f}_1\ \mathtt{x} +_{\mathbb{C}} \mathtt{f}_2\ \mathtt{x}$

Complex-valued functions (1/2)

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Operation	Notation	Definition
cfun_add	$\texttt{f}_1 +_{\texttt{cfun}} \texttt{f}_2$	$\lambda \texttt{x} : \texttt{A}.\ \texttt{f}_1\ \texttt{x} +_{\mathbb{C}} \texttt{f}_2\ \texttt{x}$
cfun_smul	$\texttt{a}\%\texttt{f}$	$\lambda \texttt{x} : \texttt{A}.\ \texttt{a} * \texttt{f}\ \texttt{x}$
cfun_neg	$-\texttt{f}$	$\lambda \texttt{x} : \texttt{A}.\ -1\%(\texttt{f}\ \texttt{x})$
cfun_sub	$\texttt{f}_1 - \texttt{f}_2$	$\texttt{f}_1 + -\texttt{f}_2$
cfun_zero		$\lambda \texttt{x} : \texttt{A}.\ 0$

Complex-valued functions (2/2)

Theorem (Complex functions are a vector space)

Addition commutativity	$\mathtt{x} + \mathtt{y} = \mathtt{y} + \mathtt{x}$
Addition associativity	$(\mathtt{x} + \mathtt{y}) + \mathtt{z} = \mathtt{x} + \mathtt{y} + \mathtt{z}$
Left distributivity	$\mathtt{a} \ \% \ (\mathtt{x} + \mathtt{y}) = \mathtt{a} \ \% \ \mathtt{x} + \mathtt{a} \ \% \ \mathtt{y}$
Identity element	$\mathtt{x} + \mathtt{cfun\_zero} = \mathtt{x}$

$+$ tactic to automatize arithmetic reasoning: `CFUN_ARITH_TAC`.
$\longrightarrow$ allows to prove many other properties.

Operators over functions

Definition (Complex-function operators type)

$\mathtt{cop} = (\mathtt{A} \to \mathtt{complex}) \to (\mathtt{B} \to \mathtt{complex})$

Definition (Algebraic operations on cop)

Operation	Notation	Definition
cop_mul	$\mathtt{op}_1 * *\mathtt{op}_2$	$\lambda \mathtt{f} : \mathtt{A} \to \mathtt{complex}.\ \mathtt{op}_1\ (\mathtt{op}_2\ \mathtt{f})$

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cop_mul	$\mathtt{op}_1 ** \mathtt{op}_2$	$\lambda \mathtt{f} : \mathtt{A} \to \mathtt{complex}.\ \mathtt{op}_1\ (\mathtt{op}_2\ \mathtt{f})$
cop_add	$\mathtt{op}_1 +_{\mathtt{cop}} \mathtt{op}_2$	$\lambda \mathtt{f} : \mathtt{A} \to \mathtt{complex}.\ \mathtt{op}_1\ \mathtt{f} +_{\mathtt{cfun}} \mathtt{op}_2\ \mathtt{f}$
cop_smul	$\mathtt{a}\ \%_{\mathtt{cop}}\ \mathtt{op}$	$\lambda \mathtt{f} : \mathtt{A} \to \mathtt{complex}.\ \mathtt{a}\ \%_{\mathtt{cfun}}\ \mathtt{op}\ \mathtt{f}$

and negation, zero, etc.

- cop_mul is not commutative.
- COP_ARITH_TAC.

Linearity

Definition

$$\texttt{is\_linear\_cop}\ (\texttt{op} : \texttt{cop}) \Leftrightarrow$$
$$\forall \texttt{f g. op}\ (\texttt{f} + \texttt{g}) = \texttt{op f} + \texttt{op g}\ \wedge \forall \texttt{a. op}\ (\texttt{a}\ \%\ \texttt{f}) = \texttt{a}\ \%\ (\texttt{op f})$$

Note: In finite dimension, linear operator are matrices.

Linearity

Definition

$$\texttt{is\_linear\_cop}\ (\texttt{op} : \texttt{cop}) \Leftrightarrow$$
$$\forall \texttt{f g. op (f + g) = op f + op g} \ \wedge \forall \texttt{a. op (a \% f) = a \% (op f)}$$

Note: In finite dimension, linear operator are matrices.

In general: $\texttt{op}_3\ **\ (\texttt{op}_1 + \texttt{op}_2) \neq \texttt{op}_3\ **\ \texttt{op}_1 + \texttt{op}_3\ **\ \texttt{op}_2$

But, for linear operators:

Theorem

$$\forall \texttt{op}_1\ \texttt{op}_2\ \texttt{op}_3.\ \texttt{is\_linear\_cop}\ \texttt{op}_3 \Rightarrow$$
$$\texttt{op}_3\ **\ (\texttt{op}_1 + \texttt{op}_2) = \texttt{op}_3\ **\ \texttt{op}_1 + \texttt{op}_3\ **\ \texttt{op}_2$$

Linearity (composition)

Composition relations:

Theorem

$$\forall \mathtt{op}_1\ \mathtt{op}_2.\ \mathtt{is\_linear\_cop}\ \mathtt{op}_1 \wedge \mathtt{is\_linear\_cop}\ \mathtt{op}_2 \Rightarrow$$
$$\mathtt{is\_linear\_cop}\ (\mathtt{op}_1 + \mathtt{op}_2) \wedge \mathtt{is\_linear\_cop}\ (\mathtt{op}_1 ** \mathtt{op}_2) \wedge$$
$$\mathtt{is\_linear\_cop}\ (\mathtt{op}_2 - \mathtt{op}_1) \wedge \forall \mathtt{a}.\ \mathtt{is\_linear\_cop}\ (\mathtt{a}\ \%\ \mathtt{op}_1)$$

$+$ tactic to automatize the proof that a function is linear: `LINEARITY_TAC`.

Note: Interaction-oriented tactic

Inner Product: Definition

Definition

$$\texttt{is\_inprod}\ (\texttt{inprod} : \texttt{cfun} \to \texttt{cfun} \to \texttt{complex}) \Leftrightarrow$$
$$\quad \forall\ \texttt{x y z}.$$
$$\qquad \texttt{cnj}\ (\texttt{inprod y x}) = \texttt{inprod x y} \wedge$$

Inner Product: Definition

Definition

$$
\begin{aligned}
&\texttt{is\_inprod}\ (\texttt{inprod} : \texttt{cfun} \rightarrow \texttt{cfun} \rightarrow \texttt{complex}) \Leftrightarrow \\
&\quad \forall\ \texttt{x}\ \texttt{y}\ \texttt{z}. \\
&\qquad \texttt{cnj}\ (\texttt{inprod}\ \texttt{y}\ \texttt{x}) = \texttt{inprod}\ \texttt{x}\ \texttt{y}\ \wedge \\
&\qquad \texttt{inprod}\ (\texttt{x} + \texttt{y})\ \texttt{z} = \texttt{inprod}\ \texttt{x}\ \texttt{z} + \texttt{inprod}\ \texttt{y}\ \texttt{z}\ \wedge
\end{aligned}
$$

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\qquad \texttt{real}\ (\texttt{inprod x x}) \wedge 0 \le \texttt{real\_of\_complex}\ (\texttt{inprod x x}) \wedge
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\qquad (\texttt{inprod x x} = 0 \Leftrightarrow \texttt{x} = \texttt{cfun\_zero}) \wedge \\
\qquad \forall \texttt{a}.\ \texttt{inprod x}\ (\texttt{a}\ \%\ \texttt{y}) = \texttt{a} * (\texttt{inprod x y})
\end{array}
$$

Note: axiomatic definition, because it depends on the type

Inner Product: Properties

Many theorems, notably:

- Orthogonal projection
- Injectivity of inner product seen as a curried function
- Pythagorean Theorem
- Cauchy-Schwarz inequality

Other notions

- Eigenvalues and eigenvectors
- Orthogonality
- Hermitian adjoint
- Self-adjoint
- $+$ tactics

Other notions

- Eigenvalues and eigenvectors
- Orthogonality
- Hermitian adjoint
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- $+$ tactics

A theorem making use of all these notions:

Theorem

$\forall\ \mathtt{inprod\ op\ f_1\ f_2\ z_1\ z_2}.$
$\quad \mathtt{is\_inprod\ inprod} \wedge$
$\quad \mathtt{is\_self\_adjoint\ op\ inprod} \wedge \mathtt{z_1} \neq \mathtt{z_2} \wedge$
$\quad \mathtt{is\_eigen\_pair\ op\ (f_1, z_1)} \wedge \mathtt{is\_eigen\_pair\ op\ (f_2, z_2)}$
$\quad\quad \Rightarrow \mathtt{are\_orthogonal\ inprod\ f_1\ f_2}$

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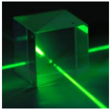
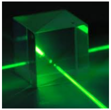
Quantum Beam Splitter



- Beam splitter = four-port optical device:
  - Two inputs = light beams.
  - Two outputs = light beams.

Quantum Beam Splitter



- Beam splitter = four-port optical device:
  - Two inputs = light beams.
  - Two outputs = light beams.
- In quantum optics:
  - Light = stream of photons.
  - Stream of photons = quantum single-mode electromagnetic field.

Single-Mode Formalization

- A single-mode emf is characterized by:
  - Its electrical charge $\hat{q}$.
  - Its flux density $\hat{p}$.
  - Its total energy: $\hat{H}(t) = \frac{\omega^2}{2}\hat{q}(t)^2 + \frac{1}{2}\hat{p}(t)^2$.

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Definition

$$\begin{array}{l}\mathtt{is\_sm}\ ((\mathtt{qs}, \mathtt{cs}, \mathtt{H}), \omega : \mathtt{sm}) \Leftrightarrow \\ \quad \mathtt{is\_qsys}\ (\mathtt{qs}, [\mathtt{p}, \mathtt{q}], \mathtt{H}) \wedge 0 < \mathtt{omega} \wedge \\ \quad \mathtt{H} = \frac{\omega^2}{2}\ \%\ (\mathtt{q}\ **\ \mathtt{q}) + \frac{1}{2}\ \%\ (\mathtt{p}\ **\ \mathtt{p})\end{array}$$

Beam Splitter Formalization

Beam splitter relates q or p of inputs, and respective outputs as follows:

$$\begin{pmatrix} q_{out_1} \\ \\ q_{out_2} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ \\ b_3 & b_4 \end{pmatrix} * \begin{pmatrix} q_{in_1} \\ \\ q_{in_2} \end{pmatrix}$$

+ similar for p

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+ similar for p

Definition (Beam Splitter)

$\texttt{is\_bmsp}\ (\texttt{b}_1, \texttt{b}_2, \texttt{b}_3, \texttt{b}_4, \texttt{in\_port}_1, \texttt{in\_port}_2, \texttt{out\_port}_1, \texttt{out\_port}_2) \Leftrightarrow$
$\texttt{is\_sm in\_port}_1 \wedge \texttt{is\_sm in\_port}_2$
$\wedge \texttt{is\_sm out\_port}_1 \wedge \texttt{is\_sm out\_port}_2$

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$+$ similar for p

Definition (Beam Splitter)

$\mathtt{is\_bmsp}\ (\mathtt{b}_1, \mathtt{b}_2, \mathtt{b}_3, \mathtt{b}_4, \mathtt{in\_port}_1, \mathtt{in\_port}_2, \mathtt{out\_port}_1, \mathtt{out\_port}_2) \Leftrightarrow$
$\mathtt{is\_sm\ in\_port}_1 \wedge \mathtt{is\_sm\ in\_port}_2$
$\wedge \mathtt{is\_sm\ out\_port}_1 \wedge \mathtt{is\_sm\ out\_port}_2$
$\wedge \mathtt{p}_{\mathtt{out}_1} = \mathtt{b}_1\ \%\ \mathtt{p}_{\mathtt{in}_1} + \mathtt{b}_2\ \%\ \mathtt{p}_{\mathtt{in}_2}\ \wedge \mathtt{q}_{\mathtt{out}_1} = \mathtt{b}_1\ \%\ \mathtt{q}_{\mathtt{in}_1} + \mathtt{b}_2\ \%\ \mathtt{q}_{\mathtt{in}_2}$
$\wedge \mathtt{p}_{\mathtt{out}_2} = \mathtt{b}_3\ \%\ \mathtt{p}_{\mathtt{in}_1} + \mathtt{b}_4\ \%\ \mathtt{p}_{\mathtt{in}_2}\ \wedge \mathtt{q}_{\mathtt{out}_2} = \mathtt{b}_3\ \%\ \mathtt{q}_{\mathtt{in}_1} + \mathtt{b}_4\ \%\ \mathtt{q}_{\mathtt{in}_2}$

Beam Splitter Energy Preservation

Main result:

Theorem (Energy Preservation)

$\forall\ \mathtt{bs.\ is\_bmsp\ bs} \Rightarrow \mathtt{H}_{\mathtt{in}_1} + \mathtt{H}_{\mathtt{in}_2} = \mathtt{H}_{\mathtt{out}_1} + \mathtt{H}_{\mathtt{out}_2}$

(note: H is the Hamiltonian, i.e. energy)

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Conclusion

- HOL Formalization of complex function spaces.
- Formalization of related concepts: linearity, inner products,. . .
- Application-oriented formalization, useful for engineering verification.
- Application to quantum theory, prove beam splitter energy preservation.
- Around 1000 lines of code with 160 theorems
  $\longrightarrow$ big code size reduction thanks to automation

Future Work

- Instantiation to finite-dimension complex vectors
  $\longrightarrow$ applications in electromagnetics and ray optics
- Advanced formalization of quantum optics
  $\longrightarrow$ quantum computers

Concordia University
Hardware Verification Group
Faculty of Engineering and Computer Science

http://hvg.ece.concordia.ca

Thanks!
Questions?

PS: Still looking for a job in Germany... :-)

Eigenvalues & Eigenvectors

Definition (Eigen pair)

$$\texttt{is\_eigen\_pair}\ (\texttt{op} : \texttt{cop})\ (\texttt{f}, \texttt{v}) \Leftrightarrow$$
$$\quad \texttt{is\_linear\_cop op} \Rightarrow \texttt{op f} = \texttt{v \% f} \ \wedge \texttt{f} \neq \texttt{zerofun}$$

$\longrightarrow$ very useful in applications

Theorem (Subspace of eigenvectors)

$$\forall \texttt{op}.\ \texttt{is\_linear\_cop op} \Rightarrow$$
$$\quad \forall \texttt{z}.\ \texttt{is\_subspace}$$
$$\qquad (\{\ \texttt{f} \mid \texttt{is\_eigen\_pair op}\ (\texttt{f}, \texttt{z})\} \cup \{\texttt{cfun\_zero}\})$$

Orthogonality

Definition (Orthogonality)

$$\texttt{are\_orthogonal inprod u v} \Leftrightarrow$$
$$\quad \texttt{is\_inprod inprod} \Rightarrow \texttt{inprod u v} = \texttt{Cx(\&0)}$$

Orthogonality

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$\texttt{are\_orthogonal inprod u v} \Leftrightarrow$
$\quad \texttt{is\_inprod inprod} \Rightarrow \texttt{inprod u v} = \texttt{Cx(\&0)}$

Many theorems, notably:

Theorem (Pythagorean Theorem)

$\forall\ \texttt{inprod u v. is\_inprod inprod} \land\ \texttt{are\_orthogonal inprod u v} \Rightarrow$
$\quad \texttt{inprod (u + v) (u + v)} = \texttt{inprod u u} + \texttt{inprod v v}$

Orthogonality

Definition (Orthogonality)

are_orthogonal inprod u v $\Leftrightarrow$
  is_inprod inprod $\Rightarrow$ inprod u v $=$ Cx(&0)

Many theorems, notably:

Theorem (Pythagorean Theorem)

$\forall$ inprod u v. is_inprod inprod $\wedge$ are_orthogonal inprod u v $\Rightarrow$
  inprod (u $+$ v) (u $+$ v) $=$ inprod u u $+$ inprod v v

Theorem (Cauchy-Schwarz inequality)

$\forall$ x y inprod. is_inprod inprod $\Rightarrow$
  norm (inprod x y) pow 2 $\leq$
    real_of_complex (inprod x x) $*$ real_of_complex (inprod y y)

Hermitian adjoint

Definition (Hermitian)

$$
\begin{array}{l}
\texttt{is\_hermitian}\ \texttt{op}_1\ \texttt{op}_2\ \texttt{inprod} \Leftrightarrow \\
\quad \texttt{is\_inprod}\ \texttt{inprod} \Rightarrow \\
\quad\quad \texttt{is\_linear\_cop}\ \texttt{op}_1 \wedge \texttt{is\_linear\_cop}\ \texttt{op}_2\ \wedge \\
\quad\quad \forall\ \texttt{x}\ \texttt{y}.\ \texttt{inprod}\ \texttt{x}\ (\texttt{op}_1\ \texttt{y}) = \texttt{inprod}\ (\texttt{op}_2\ \texttt{x})\ \texttt{y}
\end{array}
$$

Note: in finite dimension, hermitian operation = matrix conjugate transpose.

Hermitian adjoint

Definition (Hermitian)

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\texttt{is\_hermitian}\ \texttt{op}_1\ \texttt{op}_2\ \texttt{inprod} \Leftrightarrow \\
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\quad\quad \forall\ \texttt{x}\ \texttt{y}.\ \texttt{inprod}\ \texttt{x}\ (\texttt{op}_1\ \texttt{y}) = \texttt{inprod}\ (\texttt{op}_2\ \texttt{x})\ \texttt{y}
\end{array}
$$

Note: in finite dimension, hermitian operation = matrix conjugate transpose.

In general, the existence of an adjoint is not ensured
BUT, if it exists, it is unique:

Theorem (Unicity of hermitian)

$$
\begin{array}{l}
\forall \texttt{op}_1\ \texttt{op}_2\ \texttt{op}_3\ \texttt{inprod}. \\
\quad \texttt{is\_hermitian}\ \texttt{op}_1\ \texttt{op}_2\ \texttt{inprod} \wedge \texttt{is\_hermitian}\ \texttt{op}_1\ \texttt{op}_3\ \texttt{inprod} \\
\quad\quad \Rightarrow \texttt{op}_2 = \texttt{op}_3
\end{array}
$$

Self-Adjoint

Definition

is_self_adjoint op inprod $\Leftrightarrow$ is_hermitian op op inprod

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is_self_adjoint op inprod $\Leftrightarrow$ is_hermitian op op inprod

Theorem

$\forall$ inprod op x y.
  is_inprod inprod $\wedge$ is_linear_op op $\wedge$
  inprod (op x) y = $-$(inprod x (op y)))
    $\Rightarrow$ is_self_adjoint (ii % op) inprod

Self-Adjoint

Definition

is_self_adjoint op inprod $\Leftrightarrow$ is_hermitian op op inprod

Theorem

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is_inprod inprod $\wedge$ is_linear_op op $\wedge$
inprod (op x) y = −(inprod x (op y)))
$\Rightarrow$ is_self_adjoint (ii % op) inprod

Theorem

$\forall$ inprod op. is_inprod inprod $\wedge$ is_self_adjoint op inprod $\Rightarrow$
$\forall$z. is_eigen_value op z $\Rightarrow$ real z

A Theorem Making Use of All Notions

Theorem

$$\begin{array}{l}\forall\ \mathtt{inprod}\ \mathtt{op}\ \mathtt{f}_1\ \mathtt{f}_2\ \mathtt{z}_1\ \mathtt{z}_2.\\ \quad \mathtt{is\_inprod}\ \mathtt{inprod}\ \wedge\\ \quad \mathtt{is\_self\_adjoint}\ \mathtt{op}\ \mathtt{inprod}\ \wedge \mathtt{z}_1 \neq \mathtt{z}_2\ \wedge\\ \quad \mathtt{is\_eigen\_pair}\ \mathtt{op}\ (\mathtt{f}_1, \mathtt{z}_1) \wedge \mathtt{is\_eigen\_pair}\ \mathtt{op}\ (\mathtt{f}_2, \mathtt{z}_2)\\ \qquad \Rightarrow \mathtt{are\_orthogonal}\ \mathtt{inprod}\ \mathtt{f}_1\ \mathtt{f}_2\end{array}$$